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On difference equations with asymptotically stable 2-cycles perturbed by a decaying noise[☆,☆☆]

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ABSTRACT

The results stating that the stability of a 2-cycle is preserved (almost surely) under an eventually decaying stochastic perturbation are obtained in the case when the system is in the range of the parameters immediately following the Hopf bifurcation and preceding the next period doubling bifurcation. Several examples of systems and types of noise are presented.

1. Introduction

Difference equations are applied as adequate models of population dynamics, for example, for species with seasonal reproduction. There are many publications concerning stability of discrete nonlinear systems, preservation of stability under deterministic and stochastic perturbations, see, for example, recent publications [1–5] and references therein. However, oscillatory (and in particular 2-cyclic) behavior is characteristic for many real world systems, and is more frequently observed than convergence to a stable equilibrium, and sustainable oscillations cannot be explained by random noise only. Meanwhile, there are very few papers where the research is focused on stable oscillatory behavior and in particular on stabilization by a random perturbation or noise [6–11]. Also, if the cyclic behavior of the nonperturbed equation is combined with a stochastic perturbation, such systems have so far received very little attention in the literature. In [6] it is explained how random perturbations can cause blurred stable orbits in an otherwise chaotic systems, see also [7].

In the present paper we assume that the original system is in the range of parameters leading to a stable 2-cycle and deduce conditions under which the orbits of a stochastically perturbed system eventually stay in a δ-neighborhood of this 2-cycle with a probability $1-\gamma$, for any given small $\delta > 0$, $\gamma > 0$. Stochastic perturbations of stable 2-cycles are studied in the case when the range of the parameters is between the first and the second period doubling bifurcations. The well-known logistic $F_\mu(x) = \mu x(1-x)$, Ricker $F_\mu(x) = x\exp(\mu(1-x))$, Hassel and May $F_\mu(x) = \mu x(1+x)^{-d}$, $d > 1$, and Bellows maps $F_\mu(x) = \mu x(1+(ax)^d)^{-1}$, $a > 0$, $d > 1$, for appropriate values of the parameter $\mu > 0$, are examples of such systems (see more details in Section 5.1).

In this paper a scalar difference equation depending on parameter is perturbed by a vanishing stochastic noise ρ_n such that, almost surely,

$$\rho_n \to 0 \quad \text{as } n \to \infty. \tag{1}$$

The paper is organized as follows. After some preliminaries in Section 2, a scalar difference equation with an asymptotically stable 2-cycle perturbed by a decaying noise is considered in Section 3. The main result is Theorem 3.1 which claims that if

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the equation has a stable 2-cycle, then under a vanishing noise positive solutions tend to this cycle almost surely. In Section 4 we discuss how to find $j(\gamma)$ satisfying

$$\mathbb{P}\{|\rho_n| < j, \text{ for all } n \in \mathbf{N}\} > 1 - \gamma,$$

when condition (1) holds, $\rho_n = \sigma_{n-1}\xi_n$, where σ_{n-1} are nonrandom coefficients and ξ_n are random variables. We compute the values of $j(\gamma)$ for two types of σ_n and for $\gamma = 0.05$, when distributions of ξ_n have square-exponential tails. In Section 5 we describe some well-known maps along with the range of parameters which provide an asymptotically stable limit cycle. In conclusion we present a numerical example of the logistic equation perturbed by a decaying noise.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

We use the standard abbreviation "a.s." for the wordings "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbb{P}$ throughout the text. A detailed discussion of probabilistic concepts and notation may be found, for example, in [12].

Everywhere in this paper we suppose that assumption (1) holds a.s. Conditions which guarantee (1) are given in [2] (see also [4]). The next lemma is proved in [2].

Lemma 2.1. *Let (1) holds a.s. Then $\forall\gamma \in (0, 1)$ there exist $\Omega_\gamma \subseteq \Omega$ and $j(\gamma)$ such that*

$$\sup_{n\in\mathbf{N}} |\rho_n(\omega)| < j(\gamma), \quad \omega \in \Omega_\gamma,\ \mathbb{P}(\Omega_\gamma) > 1 - \gamma. \tag{2}$$

In the proof of the main theorem we will use the following elementary lemma, whose proof we present for completeness of the argument.

Lemma 2.2. *Let $k_1 \in (0, 1)$, $\beta_i \geq 0$ for each $i \in \mathbf{N}$, and $\lim_{i\to\infty} \beta_i = 0$. Then*

$$\sum_{i=0}^{n} k_1^i \beta_{n-i} \to 0, \quad \textit{as } n \to \infty.$$

Proof. Fix $\varepsilon_0 > 0$. Let $N_1 \in \mathbf{N}$ be so large that

$$\frac{\beta k_1^{N_1+1}}{(1-k_1)} < \varepsilon_0/2.$$

Suppose that $N_2 \in \mathbf{N}$ is so large that for $n \geq N_2$

$$\beta_n < \frac{\varepsilon_0(1-k_1)}{2}.$$

Then, for $n > N_2 + N_1$,

$$\begin{aligned}
\sum_{i=0}^{n} k_1^i \beta_{n-i} &= \sum_{i=0}^{N_1} k_1^i \beta_{n-i} + \sum_{i=N_1+1}^{n} k_1^i \beta_{n-i} \\
&\leq \max_{j\geq N_2}\{\beta_j\} \sum_{i=0}^{N_1} k_1^i + \beta k_1^{N_1+1} \sum_{i=0}^{\infty} k_1^i \\
&\leq \frac{\varepsilon_0(1-k_1)}{2} \frac{1}{1-k_1} + \beta k_1^{N_1+1} \frac{1}{1-k_1} < \varepsilon_0. \quad \square
\end{aligned}$$

3. Main result

Let $F_\mu : \mathbf{R} \to \mathbf{R}$ be continuously differentiable for $\mu \in (\bar{\mu}, \hat{\mu})$. We define

$$F_\mu^2(x) := F_\mu(F_\mu(x)), \tag{3}$$

and note that F_μ^2 is also continuously differentiable on $\mathbf{R}$.

Suppose that the equation

$$x_{n+1} = F_\mu(x_n), \quad n \in \mathbf{N}, \tag{4}$$

has a 2-cycle, $\{x_\mu(0), x_\mu(1)\}$, when $\mu \in (\mu_1, \mu_2) \subset (\bar{\mu}, \hat{\mu})$. Moreover, suppose that for each $\mu \in (\mu_1, \mu_2)$

$$\left|(F_\mu^2)'(x_\mu(0))\right| = \left|F_\mu'(x_\mu(0))\right| \left|F_\mu'(x_\mu(1))\right| = k_\mu < 1. \tag{5}$$

Condition (5) implies that a 2-cycle, $\{x_\mu(0), x_\mu(1)\}$, is asymptotically stable for each $\mu \in (\mu_1, \mu_2)$ (see [13, Theorem 1.22, page 39]).

We consider the stochastically perturbed difference equation

$$X_{n+1} = F_\mu(X_n) + \varepsilon\rho_{n+1}, \quad n \in \mathbf{N}, \tag{6}$$

where ρ_n are random variables satisfying (1), $\varepsilon > 0$ is a small parameter, $\mu \in (\mu_1, \mu_2)$.

Theorem 3.1. *Let F_μ^2 be continuously differentiable on $\mathbf{R}$ for each $\mu \in (\mu_1, \mu_2)$, condition (5) hold, $(\rho_n)_{n\in\mathbf{N}}$ be a sequence of random variables satisfying (1), and $\{x_\mu(0), x_\mu(1)\}$ be an asymptotically stable 2-cycle for Eq. (4).*

Then for each $\mu \in (\mu_1, \mu_2)$ and $\gamma \in (0, 1)$ there exist $\delta_1 = \delta_1(\mu) > 0$, $\Omega_\gamma \subseteq \Omega$ with $\mathbb{P}(\Omega_\gamma) > 1 - \gamma$, and $\varepsilon_0 = \varepsilon_0(\mu, \gamma) > 0$ such that for all $\omega \in \Omega_\gamma$ and $\varepsilon < \varepsilon_0$ solution $X_n(\omega)$ of (6) with any initial value $X_0 \in (x_\mu(0)-\delta_1, x_\mu(0)+\delta_1)$ satisfies

$$X_{2k}(\omega) \in (x_\mu(0) - \delta_1, x_\mu(0) + \delta_1), \quad \text{for each } k \in \mathbf{N}, \tag{7}$$

and

$$\lim_{k\to\infty} X_{2k} = x_\mu(0), \qquad \lim_{k\to\infty} X_{2k+1} = x_\mu(1) \quad a.s.$$

Proof. We fix some $\mu \in (\mu_1, \mu_2)$. Due to continuous differentiability of F_μ^2 there exist $\delta_1 \in (0, 1)$ and $k_1 \in (k_\mu, 1)$, where k_μ is defined in (5), such that for $x \in (x_\mu(0) - \delta_1, x_\mu(0) + \delta_1)$

$$\left|(F_\mu^2)'(x)\right| < k_1 < 1. \tag{8}$$

Let $x \in (x_\mu(0) - \delta_1, x_\mu(0) + \delta_1)$ and $\theta \in (0, 1)$. Let $\theta_1, \theta_2 \in (-\theta, \theta)$; then

$$F_\mu\big(F_\mu(x) + \theta_1\big) + \theta_2 - F_\mu^2(x) = F_\mu(F_\mu(x)) + F'_\mu(\varsigma)\theta_1 + \theta_2 - F_\mu^2(x) = F'_\mu(\varsigma)\theta_1 + \theta_2, \tag{9}$$

where ς is a value between $F_\mu(x)$ and $F_\mu(x) + \theta_1$. We put

$$H_\mu^{(1)} := \min_{x\in(x_\mu(0)-\delta_1, x_\mu(0)+\delta_1)} \{F_\mu(x)\}, \qquad H_\mu^{(2)} := \max_{x\in(x_\mu(0)-\delta_1, x_\mu(0)+\delta_1)} \{F_\mu(x)\}$$

and

$$H_\mu^{(3)} := \max_{x\in(H_\mu^{(1)}-1, H_\mu^{(2)}+1)} |F'_\mu(x)|.$$

Then, for $x \in (x_\mu(0) - \delta_1, x_\mu(0) + \delta_1)$, we have

$$|F_\mu\big(F_\mu(x) + \theta_1\big) + \theta_2 - F_\mu^2(x)| \le |F'_\mu(\varsigma)|\,|\theta_1| + |\theta_2| \le |\theta|\left[H_\mu^{(3)} + 1\right]. \tag{10}$$

Further, for $x_0 \in (x_\mu(0) - \delta_1, x_\mu(0) + \delta_1)$,

$$\begin{aligned} |F_\mu\big(F_\mu(x_0) + \theta_1\big) + \theta_2 - x_\mu(0)| &= |F_\mu\big(F_\mu(x_0) + \theta_1\big) + \theta_2 - F_\mu^2(x_\mu(0))| \\ &= |F_\mu\big(F_\mu(x_0) + \theta_1\big) + \theta_2 - F_\mu^2(x_0)| + |F_\mu^2(x_\mu(0)) - F_\mu^2(x_0)| \\ &\le |\theta|\left[H_\mu^{(3)} + 1\right] + k_1|x_\mu(0) - x_0| \\ &\le |\theta|\left[H_\mu^{(3)} + 1\right] + k_1\delta_1. \end{aligned} \tag{11}$$

So, in order to have

$$|\theta|\left[H_\mu^{(3)} + 1\right] + k_1\delta_1 < \delta_1,$$

θ has to satisfy

$$|\theta| < \frac{\delta_1(1 - k_1)}{\left[H_\mu^{(3)} + 1\right]}. \tag{12}$$

Fix $\gamma \in (0, 1)$. Applying Lemma 2.1 we can find $\Omega_\gamma \subseteq \Omega$ and $j(\gamma)$ such that

$$\sup_{n\in\mathbf{N}} |\rho_n(\omega)| < j(\gamma), \quad \omega \in \Omega_\gamma,\ \mathbb{P}(\Omega_\gamma) > 1 - \gamma. \tag{13}$$

Fix $\omega \in \Omega_\gamma$ and put

$$\theta_1 = \varepsilon\rho_1(\omega), \qquad \theta_2 = \varepsilon\rho_2(\omega),$$

with

$$\varepsilon < \frac{\delta_1(1-k_1)}{\left[H_\mu^{(3)}+1\right]j(\gamma)}. \tag{14}$$

Then, for $\omega \in \Omega_\gamma$, estimation (11) implies that

$$\begin{aligned} |X_2(\omega) - x_\mu(0)| &= |F_\mu\big(F_\mu(X_0(\omega) + \varepsilon\rho_1(\omega))\big) + \varepsilon\rho_2(\omega) - x_\mu(0)| \\ &= |F_\mu\big(F_\mu(X_0(\omega)) + \theta_1\big) + \theta_2 - x_\mu(0)| < \delta_1, \end{aligned}$$

i.e. $X_2(\omega) \in (x_\mu(0) - \delta_1, x_\mu(0) + \delta_1)$. Applying mathematical induction in a similar manner, we conclude that $X_{2k}(\omega) \in (x_\mu(0) - \delta_1, x_\mu(0) + \delta_1)$ for each $k \in \mathbf{N}$.

Let $\omega \in \Omega_\gamma$, $x \in (x_\mu(0) - \delta_1, x_\mu(0) + \delta_1)$, $|\theta_1|, |\theta_2| < 1$ and let $n \in \mathbf{N}$ be arbitrary. We denote

$$\mathcal{A}(x, \theta_1, \theta_2) := F_\mu\big(F_\mu(x) + \theta_1\big) + \theta_2 - F_\mu^2(x)$$

and

$$\mathcal{A}(n, \omega) := \mathcal{A}(X_{2n}(\omega), \varepsilon\rho_{2n+1}(\omega), \varepsilon\rho_{2(n+1)}(\omega)).$$

We note by (10) that

$$|\mathcal{A}(n, \omega)| \le H_\mu^{(3)}|\varepsilon\rho_{2n+1}(\omega)| + \varepsilon|\rho_{2(n+1)}(\omega)| := \alpha_{2n}$$

and for some $\alpha > 0$, for all $\omega \in \Omega_\gamma$, $n \in \mathbf{N}$,

$$\alpha_{2n} < \alpha, \qquad \alpha_{2n} \to 0, \quad \text{as } n \to \infty.$$

Denote $\Delta_n(\omega) = |X_{2n}(\omega) - x_\mu(0)|$. Then, recalling that $x_\mu(0) = F_\mu^2(x_\mu(0))$, we estimate for $\omega \in \Omega_\gamma$ and $x_0 \in (x_\mu(0) - \delta_1, x_\mu(0) + \delta_1)$

$$\begin{aligned} \Delta_{n+1}(\omega) &= |X_{2(n+1)}(\omega) - x_\mu(0)| = |F_\mu^2\,(X_{2n}(\omega)) - F_\mu^2(x_\mu(0)) + \mathcal{A}(n, \omega)| \\ &\le |F_\mu^2\,(X_{2n}(\omega)) - F_\mu^2(x(0))| + |\mathcal{A}(n, \omega)| \\ &\le k_1|X_{2n}(\omega) - x_\mu(0)| + \alpha_{2n} = k_1\Delta_n(\omega) + \alpha_{2n}. \end{aligned}$$

Thus by induction we have

$$\Delta_{n+1}(\omega) \le k_1^{n+1}|x_0 - x_\mu(0)| + \sum_{i=0}^{n} k_1^i \alpha_{2(n-i)}.$$

Lemma 2.2 implies that $\sum_{i=0}^{n} k_1^i \alpha_{2(n-i)} \to 0$ as $n \to \infty$, and, therefore, for each $\omega \in \Omega_\gamma$,

$$X_{2n}(\omega) \to x_\mu(0).$$

To conclude the proof we note that, when $\omega \in \Omega_\gamma$,

$$X_{2n+1}(\omega) = F_\mu(X_{2n}(\omega)) + \rho_{2n+1} \to F_\mu(x_\mu(0)) = x_\mu(1)$$

as $n \to \infty$. □

Remark 3.2. Note that the first statement of the theorem, which is relation (7), holds in the case when perturbations ρ_n are not necessarily decaying but are only uniformly bounded.

4. Estimates for the bound $j(\gamma)$ of the diffusion

In this section we assume that $\rho_n = \sigma_{n-1}\xi_n$, $n \in \mathbf{N}$, where $(\sigma_n)_{n\in\mathbf{N}}$ is a sequence of nonrandom nonnegative coefficients and $(\xi_n)_{n\in\mathbf{N}}$ is a sequence of independent identically distributed continuous random variables with distribution function Ψ, zero mean and unit variance. We suppose that condition (1) holds. Without loss of generality we can also assume that σ_n are uniformly bounded.

4.1. General estimates

It is convenient for us to represent the tails of the distribution in the form of an exponent. In other words, for some $c^* > 0$, let

$$p(c) = -\ln[1 - \Psi(c) + \Psi(-c)], \quad c > c^*,$$

then

$$1 - \Psi(c) + \Psi(-c) = e^{-p(c)}, \quad c > c^*.$$

Therefore, for $c > c^*$,

$$\mathbb{P}\{|\xi_n| > c\} = 1 - \Psi(c) + \Psi(-c) = e^{-p(c)}.$$

We fix some $j > \sup_{i \in \mathbb{N}} c^* \sigma_i$, and estimate

$$\begin{aligned} \mathbb{P}\{|\sigma_n \xi_{n+1}| > j, \text{ for some } n \in \mathbf{N}\} &= \mathbb{P}\left\{|\xi_{n+1}| > \frac{j}{\sigma_n}, \text{ for some } n \in \mathbf{N}\right\} \\ &\le \mathbb{P}\left[\bigcup_{i=1}^{\infty}\left\{|\xi_{i+1}| > \frac{j}{\sigma_i}\right\}\right] \\ &\le \sum_{i=1}^{\infty} \mathbb{P}\left\{|\xi_{i+1}| > \frac{j}{\sigma_i}\right\} \\ &= \sum_{i=1}^{\infty} e^{-p(j/\sigma_i)}. \end{aligned}$$

Then

$$\mathbb{P}\{|\sigma_n \xi_{n+1}| < j, \text{ for all } n \in \mathbf{N}\} \ge 1 - \sum_{i=1}^{\infty} e^{-p(j/\sigma_i)}. \tag{15}$$

The inequality

$$\mathbb{P}\{|\sigma_n \xi_{n+1}| < j, \text{ for all } n \in \mathbf{N}\} > 1 - \gamma \tag{16}$$

holds when there exists j so big that

$$\sum_{i=1}^{\infty} e^{-p(j/\sigma_i)} < \gamma. \tag{17}$$

In this case we denote

$$j(\gamma) = \max\left[\inf\left\{H : \sum_{i=1}^{\infty} e^{-p(H/\sigma_i)} < \gamma,\right\}, \sup_{i \in \mathbb{N}} c^* \sigma_i\right] \tag{18}$$

and

$$\Omega_\gamma = \{\omega \in \Omega : |\sigma_n \xi_{n+1}| < j(\gamma), \text{ for all } n \in \mathbf{N}\}. \tag{19}$$

So, both $j(\gamma)$ and Ω_γ, defined by (18) and (19), satisfy condition (2) of Lemma 2.1.

In the next subsection we consider distributions with the square-exponential tails and calculate $j(\gamma)$ for two types of σ.

4.2. Square-exponential tails

Suppose there exists $c^* > 0$ such that the distribution Ψ of each ξ_n satisfies

$$1 - \Psi(c) + \Psi(-c) = e^{-bc^2}, \quad c \ge c^*. \tag{20}$$

This corresponds to $p(c) = bc^2$, where $b > 0$. It was proved in [2] that when the distribution of each ξ_n satisfies (20) and σ_n is monotone decreasing, $\lim_{n\to\infty} \sigma_n \xi_{n+1} = 0$ a.s. if and only if

$$\lim_{n\to\infty} \sigma_n^2 \ln n = 0. \tag{21}$$

To find $j(\gamma)$ when Ψ satisfies (20) we split $J(K) = \sum_{i=1}^{\infty} \exp\left\{-p\left(\frac{j}{\sigma_i}\right)\right\}$ into the sum of two terms:

$$J(K) = J_1(K, j) + J_2(K, j),$$

where

$$J_1(K, j) := \sum_{i=1}^{K} \exp\left\{-p\left(\frac{j}{\sigma_i}\right)\right\}, \qquad J_2(K, j) := \sum_{i=K+1}^{\infty} \exp\left\{-p\left(\frac{j}{\sigma_i}\right)\right\}.$$

Fix $\epsilon > 0$ and $j_0 > 0$. First, we need to find $K \in \mathbf{N}$ so that $J_2(K, j) < \gamma - \epsilon$ for all $j > j_0$. Next, we have to determine $j_1 > j_0$ such that for all $j > j_1$ we have $J_1(K, j) < \epsilon$.

Example 4.1. Let $\gamma = 0.05$, $p(x) = x^2$ and $\sigma_n = \ln^{-(1+\delta)}(n+1)$. Then (21) holds and

$$\begin{aligned}\exp\left\{-p\left(\frac{j}{\sigma_i}\right)\right\} &= \exp\left\{-\left(\frac{j}{\sigma_i}\right)^2\right\} \\ &= \exp\left\{-j^2 \ln^{2(1+\delta)}(i+1)\right\} \\ &= \exp\left\{\ln(i+1)(-j^2 \ln^{1+2\delta}(i+1))\right\} \\ &= (i+1)^{-j^2 \ln^{1+2\delta}(i+1)}.\end{aligned}$$

Without loss of generality we may suppose that

$$j > 3, \qquad i \ge 2, \quad 1 + 2\delta < 2.$$

Thus $j_0 = 3$ and $K = 2$. We estimate

$$j^2 \ln^{1+2\delta}(i+1) > 9 \ln 3 > 9, \quad (i+1)^{-j^2 \ln^{1+2\delta}(i+1)} < (i+1)^{-9},$$

and, therefore,

$$J_2(K, j) = J_2(2, j) = \sum_{i=3}^{\infty}(i+1)^{-j^2 \ln^{1+2\delta}(i+1)} \le \int_2^{\infty} x^{-9} dx = \frac{2^{-8}}{8} \approx 0.00049.$$

Also,

$$J_1(2, j) = \sum_{1}^{2}(i+1)^{-j^2 \ln^{1+2\delta}(i+1)} \le \sum_{1}^{2}(i+1)^{-j^2 \ln^2 2} = \left(2^{-j^2 \ln^2 2} + 3^{-j^2 \ln^2 2}\right) \le 2 \times 2^{-j^2 \ln^2 2}.$$

We want to find j such that $2 \times 2^{-j^2 \ln^2 2} < 0.049$. We estimate

$$\begin{aligned}&2^{j^2 \ln^2 2} > \frac{2}{0.049} = 40.82, \quad j^2 \ln^2 2 > \frac{\ln(40.82)}{\ln 2} \approx \frac{3.71}{0.69} \approx 5.38, \\ &j^2 > \frac{5.38}{\ln^2 2} \approx 11.19, \quad j > 3.35.\end{aligned}$$

Thus we can take $j(0.05) = 3.35$ and

$$\begin{aligned}J(3.35) &\le \sum_{i=1}^{\infty} e^{-p\left(\frac{j}{\sigma_i}\right)} \\ &= \sum_{i=1}^{2}(i+1)^{-j^2 \ln^{1+2\delta}(i+1)} + \sum_{i=3}^{\infty}(i+1)^{-j^2 \ln^{1+2\delta}(i+1)} \\ &\le 0.00049 + 0.049 < 0.05.\end{aligned}$$

In other words,

$$\mathbb{P}\{\ln^{-(1+\delta)}(n+1)|\xi_{n+1}| < 3.35, \text{ for all } n \in \mathbf{N}\} > 0.95.$$

Example 4.2. Let $\gamma = 0.05$, $p(x) = x^2$ and $\sigma_n = (n+1)^{-0.5}$. Then (21) holds and

$$\exp\left\{-p\left(\frac{j}{\sigma_i}\right)\right\} = \exp\left\{-\left(\frac{j}{\sigma_i}\right)^2\right\} = e^{-j^2(i+1)}.$$

Without loss of generality we may suppose that

$$j > \sqrt{2}, \qquad i \ge 1.$$

Thus $j_0 = \sqrt{2}$ and $K = 1$ and

$$e^{-j^2(i+1)} \le e^{-2(i+1)}.$$

Therefore

$$J_2(K,j) = J_2(1,j) = \sum_{i=2}^{\infty} e^{-j^2(i+1)} \le \sum_{i=2}^{\infty} e^{-2(i+1)} \le \int_1^{\infty} e^{-2(x+1)}dx = \frac{1}{2}e^{-4} \approx 0.009.$$

Also,

$$J_1(1,j) = e^{-2j^2}.$$

We want to find j such that $e^{-2j^2} < 0.041$, i.e. $j^2 > \frac{\ln 0.041}{-2} \approx 1.597$. Thus we can take $j(0.05) = \sqrt{2}$ and conclude that

$$\mathbb{P}\{|(n+1)^{-0.5}\xi_{n+1}| < \sqrt{2}, \text{ for all } n \in \mathbf{N}\} > 0.95.$$

5. Examples

In Section 5.1 we list some well-known maps along with the range of parameters which provide an asymptotically stable limit cycle. In items (ii)–(iv) the left end of the interval for the bifurcation parameter μ can be obtained analytically while the right one is estimated numerically.

In Section 5.2 we consider numerical examples of the logistic equation perturbed by decaying noise accompanied by computer simulations.

5.1. Examples of well-known unperturbed maps

Example 5.1. (i) *Logistic*: $F_\mu(x) = \mu x(1-x)$ has the only equilibrium point $x^* = 1 - 1/\mu$ for $\mu \in (1, \infty)$, which is stable for $1 < \mu < 2$. At $\mu = 2$ a period doubling bifurcation occurs, and for $\mu \in (\mu_1, \mu_2)$, $\mu_1 = 2$, $\mu_2 = 1 + \sqrt{6}$, there is an asymptotically stable 2-cycle $\{x_\mu(0), x_\mu(1)\}$ since condition (5) holds. At $\mu = 1 + \sqrt{6}$ another period-doubling bifurcation occurs, with a stable 4-cycle [13]. For $\mu \in (\mu_1, \mu_2)$, it is easy to demonstrate that for any initial condition $x(0) \in (0, 1)$ such that neither of its iterates coincides with the equilibrium $x^* = 1 - 1/\mu$, the solution tends to the 2-cycle $\{x_\mu(0), x_\mu(1)\}$. However, there is an infinite number of points $x(0)$ such that $F_\mu^n(x(0)) = x^*$; excluding them, we obtain the set of $x(0)$ for which $x(n)$ tends to the 2-cycle. For stochastic simulations, we should also exclude values close to 0 and 1, to avoid negative values of $x(n)$ at certain steps in the beginning. Our further examples and simulations will be for $\mu = 3.3 \in (2, 1 + \sqrt{6})$.

(ii) *Ricker*: $F_\mu(x) = xe^{\mu(1-x)}$ has the only positive equilibrium $x^* = 1$ for any $\mu > 0$. Since $F'_\mu(1) = 1 - \mu = -1$ for $\mu = 2$, the Ricker map has the first period-doubling bifurcation at $\mu_1 = 2$. If $\mu \in (\mu_1, \mu_2)$, where $\mu_2 \approx 2.5$, there is an asymptotically stable positive 2-cycle, $\{x_\mu(0), x_\mu(1)\}$, and condition (5) holds.

(iii) *Hassel and May*: $F_\mu(x) = \mu x(1+x)^{-6}$ has the only positive equilibrium $x^* = \mu^{1/6} - 1$ for $\mu = (1, \infty)$. Since $F'_\mu(x^*) = -5 + 6\mu^{-1/6} = -1$ for $\mu_1 = 1.5^6 \approx 11$, there is the first period-doubling bifurcation. The Hassel and May map has an asymptotically stable positive 2-cycle for $\mu \in (\mu_1, \mu_2)$, $\mu_2 \approx 27$.

(iv) *Bellows*: $F_\mu(x) = \mu x(1+x^6)^{-1}$ has the only positive equilibrium $x^* = (\mu - 1)^{1/6}$ for $\mu = (1, \infty)$. We have $F'_\mu(x^*) = (6 - 5\mu)/\mu = -1$ for $\mu_1 = 1.5$. The Bellows map has an asymptotically stable positive 2-cycle for $\mu \in (\mu_1, \mu_2)$, $\mu_2 \approx 1.74$.

Remark 5.2. All the maps in (i)–(iv) correspond to unimodal functions. However, the logistic map assumes initial values in (0, 1), while in (ii)–(iv) any positive initial value leads to a positive solution. Nevertheless, for simulations of stochastic perturbations with zero mean, very large (and indeed very small) initial conditions can lead to the simulation values becoming negative.

Remark 5.3. So far we considered stable 2-cycles which occur between the first and the second period doubling bifurcations. However, it would be interesting to study stochastic perturbations of stable 2-cycles for positively perturbed maps with large values of bifurcation parameter (see [8]). In particular, can a stochastic noise with a positive mean lead to blurred stable 2-cycles? This was established in [6] for the random perturbation of the Ricker model, which took one of k positive discrete values with some probabilities.

5.2. Estimations for a logistic equation perturbed by decaying noise

Consider the logistic map, $F_\mu(x) = \mu x(1-x)$, and take $\mu = 3.3$. By [13, page 45], 2-cycle $\{x_\mu(0), x_\mu(1)\}$ can be calculated by

$$x_\mu(0) = \frac{1}{2\mu}\left[(1+\mu) - \sqrt{(\mu-3)(\mu+1)}\right], \qquad x_\mu(1) = \frac{1}{2\mu}\left[(1+\mu) + \sqrt{(\mu-3)(\mu+1)}\right],$$

which gives us $x_{3.3}(0) \approx 0.48$.

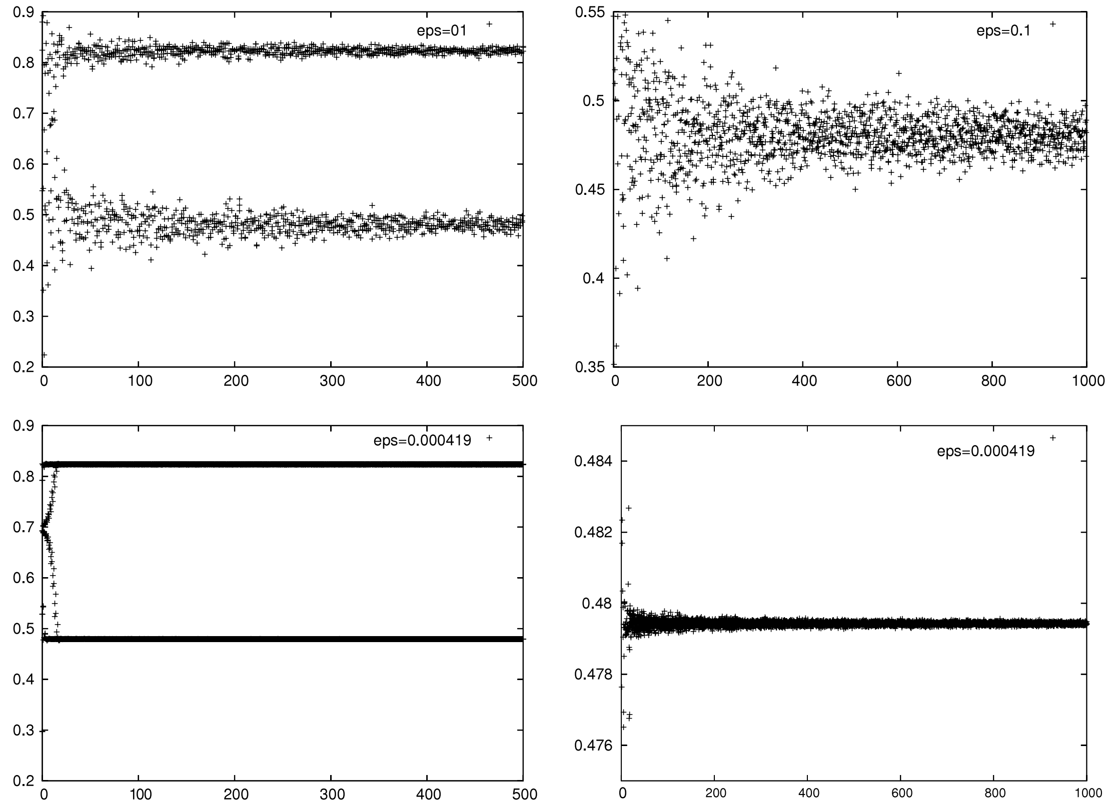

Fig. 1. Numerical simulations for the logistic equation perturbed by the stochastic noise $\varepsilon/\sqrt{n+1}\xi_{n+1}$, where $\xi_n \sim \mathcal{N}(0,1)$ for $\varepsilon = 0.1$ (top) and $\varepsilon = 0.000419$ (bottom) lead to 2-cycles. The right figures expose the lower branches of the 2-cycles only, where $\varepsilon = 0.1$ and $\varepsilon = 0.000419$, respectively.

Let the noise term be as in Example 4.2. We will find the level of admissible noise ε guaranteed by Theorem 3.1. The results of numerical simulations are demonstrated in Fig. 1.

Let $\delta_1 = 0.01$. To find k_1, defined in (8) we estimate:

$$\begin{aligned}
\max_{x\in[0.47;049]} |(F_\mu^2)'(x)| &= \max_{x\in[0.47;049]} |\mu(1-2x)\mu(1-2\mu x(1-x))| \\
&= \mu^2 \max_{x\in[0.47;049]} |(1-2x)(1-2\mu x+2\mu x^2)| \\
&\le 3.3^2 \max_{x\in[0.47;049]} \{1-2x\} \max_{x\in[0.47;0.49]} \{1-2\mu x+2\mu x^2\} \\
&= 10.89 \times 0.06 \times \left(1-2\mu \times 0.49 + 2\mu \times 0.49^2\right) \\
&\approx 10.89 \times 0.06 \times 0.65 < 0.43.
\end{aligned}$$

So we can take

$$k_1 = 0.43.$$

Now,

$$\begin{aligned}
H_\mu^{(1)} &= \min_{x\in[0.47;0.49]} \{F_\mu(x)\} = \min_{x\in[0.47;049]} \{3.3x(1-x)\} = 3.3 \times 0.47 \times 0.53 = 0.82203, \\
H_\mu^{(2)} &= \max_{x\in[0.47;0.49]} \{F_\mu(x)\} = \max_{x\in[0.47;049]} \{3.3x(1-x)\} = 3.3 \times 0.49 \times 0.51 = 0.82467, \\
H_\mu^{(3)} &= \max_{x\in[H_\mu^{(1)}-1;H_\mu^{(2)}+1]} \{|F_\mu'(x)|\} = \max_{x\in[-0.17797;1.82467]} \{3.3|1-2x|\} \approx 8.7428.
\end{aligned}$$

So ε, found by (14) can be estimated

$$\varepsilon < \frac{0.01 \times (1-0.43)}{(8.7428+1)\sqrt{2}} \approx 0.000415. \tag{22}$$

To justify the above estimation for the noise level ε, we simulate solutions of the equation

$$X_{n+1} = \mu X_n(1 - X_n) + \varepsilon\sigma_n\xi_{n+1}, \quad n \in \mathbf{N}, \tag{23}$$

where $\mu = 3.3$, $\sigma_n = (n + 1)^{-0.5}$, $\xi_n \sim \mathcal{N}(0, 1)$, $\varepsilon = 0.000419$. For comparison, we also present the graphs for $\varepsilon = 0.1$, see Fig. 1. The graph demonstrates that theoretical calculations of ε are quite conservative.

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